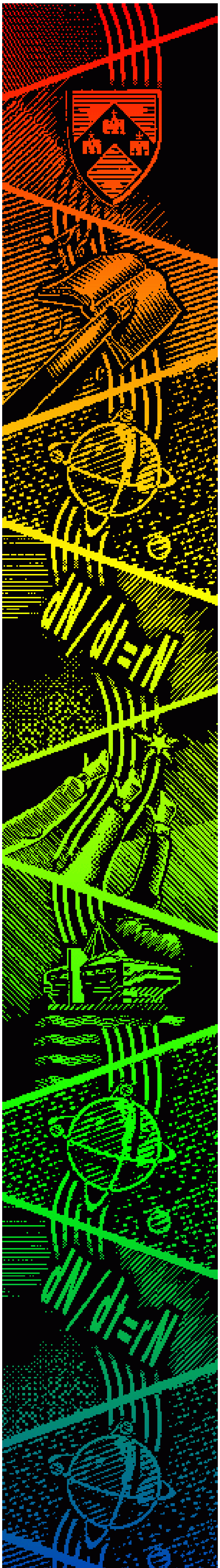

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# A Theorem on the Decentralization of the Objective Function of the Firm in GEI

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# A Theorem on the Decentralization of the Objective Function of the Firm in GEI

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**Abstract**

This paper introduces a separation theorem for a two period general equilibrium model with endogenized asset structures and incomplete income transfer space. This theorem separates the activities of the consumers from the activities of the firms. The result improves on the objective function introduced in Drèze (1974) and Grossman and Hart (1979), which is not independent of an exogenously assigned present value derived from the utilities of the owners of the firm.

Keywords: Objective Function, Incomplete Markets, Profit Maximization
JEL Classification: D20, D21, D52

## 1 Introduction

The seminal research program on the objective function of the firm in general equilibrium with incomplete markets is concentrated around the question: Do spot and equity prices provide firms with enough information to deduce what the appropriate objective function of the firm should be? The literature on the classical GEI model with production replies no to this question. It then adds the suffix, not without further extra information ([3],[5],[6],[2],[4], and others). This additional information comes from the group of owners of the firm in form of an average present value derived from the utilities of the share holders. This implies the centralization of the objective functions of the firms.

Stiefenhofer [7] initiates another research program on the objective function of the firm in general equilibrium with incomplete markets by asking, what role do intermediate goods and financial assets play in determining the production set of the firm. He considers, at variance with the two period fixed production sets of classical GEI models, one period endogenized productions sets. In this more realistic economic scenario, production sets are not independent of the financial activities of the firm. This allows to consider a new class of objective functions.

This paper shows that the extensive form of the two argument, sequential objective function of the firm introduced in Stiefenhofer [7] has a nice property, namely,

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the independence of any average utility of the owners of the firm. This property separates the activities of the agents as consumers and producers. The result is interesting because it rehabilitates the profit maximization criterion of the Arrow-Debreu model for the more general economic scenario when markets are incomplete. It essentially follows from the way capital and financial assets enter the model.

The paper is organized in four parts. Section two introduces the model. Section three states the separation theorem. Section four is a conclusion.

# 2 The Model

We consider a two period $t \in T = \{0, 1\}$ model with technological uncertainty in period 1 represented by states of nature. An element in the set of mutually exclusive and exhaustive uncertain events is denoted $s \in \{1, ..., S\}$, where by convention $s = 0$ represents the certain event in period 0. We count in total $(S+1)$ states of nature.

The economic agents are the $j \in \{1, ..., n\}$ producers and $i \in \{1, ..., m\}$ consumers which are characterized by sets of assumptions of smooth economies introduced below. There are $k \in \{1, ..., l\}$ physical commodities and $j \in \{1, ..., n\}$ financial assets, referred to as stocks. In fact, stocks are the only financial assets considered here. Physical goods are traded on each of the $(S+1)$ spot markets. Producers issue stocks which are traded at $s = 0$, yielding a payoff in the next period at uncertain state $s \in \{1, ..., S\}$. The quantity of stocks issued by firm $j \in \{1, ..., n\}$ is denoted $z_j \in \mathbb{R}_-$, where $\sum_{j=1}^{n} z_j = \hat{z}$.

There are in total $l(S+1)$ physical goods available for consumption. The consumption bundle of agent $i \in \{1, ..., m\}$ is denoted $x_i = (x_i(0), x_i(s), ..., x_i(S)) \in \mathbb{R}_{++}^{l(S+1)}$, with $x_i(s) = (x_i^1(s), ..., x_i^l(s)) \in \mathbb{R}_{++}^l$, and $\sum_{i=1}^{m} x_i = x$. The consumption space for each consumer $i \in \{1, ..., m\}$ is $X_i = \mathbb{R}_{++}^{l(S+1)}$, the strictly positive orthant. The associated price system is a collection of vectors represented by $p = (p(0), p(s), ..., p(S)) \in \mathbb{R}_{++}^{l(S+1)}$, with $p(s) = (p^1(s), ..., p^l(s)) \in \mathbb{R}_{++}^l$, the strictly positive orthant. Each consumer $i \in \{1, ..., m\}$ is endowed with initial resources $\omega_i \in \Omega$, where $\Omega = \mathbb{R}_{++}^{lT}$, and $\omega_i = (\omega_i(0), \omega_i(1))$ a collection of strictly positive vectors. Denote an initial resource vector at time period $t \in T = \{0, 1\}$, $\omega_i(t) = (\omega_i^1(t), ..., \omega_i^l(t)) \in \mathbb{R}_{++}^l$, and the sum of total initial resources, $\sum_{i=1}^{m} \omega_i = \omega$.

In total, there are $n$ financial assets traded in period $t = 0$. Denote the quantity vector of stocks purchased by consumer $i \in \{1, ..., m\}$, $z_i = (z_i(1), ..., z_i(n)) \in \mathbb{R}_+^n$, a collection of quantities of stocks purchased from producers $j \in \{1, ..., n\}$, and denote $\sum_{i=1}^{m} z_i = z$, with associated stock price system $q = (q(1), ..., q(n)) \in \mathbb{R}_{++}^n$. Denote producer $j$'s period one vector of capital purchase $y^j(0) \in \mathbb{R}_-^l$, and denote his period two state dependent net activity vector $y_j(s) = (y_j^1(s), ..., y_j^l(s)) \in \mathbb{R}^l$. Let $y_j(t = 1) = (y_j(s), ..., y_j(S)) \in \mathbb{R}^{lS}$ denote the collection of state dependent period t=1 net activity vectors. A period two input of production for every $s \in \{1, ..., S\}$ is by convention denoted $y_j^k(s) < 0$, and a production output in state $s \in \{1, ..., S\}$ satisfies $y_j^k(s) \geq 0$.

## 2.1 The model of the firm

Each firm $j \in \{1, ..., n\}$ issues stocks $z_j$ at stock price $q_j$ in period one in order to build up production capacity. A firm's total cash acquired via stock market determines the upper bound of the total value of production capacity it can install in the same period. Denote this liquidity constraint $q_j z_j = M_j$, where $M_j \in \mathbb{R}_+$ is a non-negative real number and $z_j \in \mathbb{R}_-$ a feasible financial policy of the firm $j \in \{1, ..., n\}$. $M_j$ constraints the quantity of capital $y(0) \in \mathbb{R}^l_-$ a producer $j$ can purchase at spot price system $p(0) \in \mathbb{R}^l_{++}$. The quantity of intermediate goods $y_j(0)$ purchased in period one determines a correspondence $\phi_j|_Z$. This correspondence defines the technology of the firm at feasible financial policy Z. Let the production set available to each producer $j \in \{1, ..., n\}$ in period two be described by this technology, $\phi_j|_Z : \mathbb{R}^m_- \to \mathbb{R}^n_-$, a correspondence defined on the set of period two inputs, and denote it $Y_j|_z \subset \mathbb{R}^l$. Let $S$ denote the set of all exogenously given states of nature. Then for each producer $j \in \{1, ..., n\}$ let the $t = 1$ one period production set be defined by a map $\Phi_j|_Z$ with domain $\mathbb{R}^m_- \times \mathbb{R}_{++}$ and range $\mathbb{R}^n_- \times \mathbb{R}_{++}$, and denote it $Y_j|_z(s) \subset \mathbb{R}^{l(S+1)}$, where $m + n = l$.

**Assumption 1 (Technological Uncertainty)** *For the production set of each firm $j \in \{1, ..., n\}$, denoted $Y_j|_z(s) \subset \mathbb{R}^{l(S+1)}$, the number of uncertain states of the world satisfies $S \geq 2$.*

**Assumption 2 (Smooth Endogenous Production Structures)** *Each production set $Y_j|_z(s) \subset \mathbb{R}^{l(S+1)}$ for $j \in \{1, ..., n\}$, satisfies the differentiability properties for smooth economies.*[1]

Using assumption 1 (Technological Uncertainty), the implicit assumption of long run profits maximization, and assumption 2 (Smooth Production Sets), a closed form objective function can be assigned to each producer $j \in \{1, ..., n\}$. The algebraic form of the long run profit maximization objective function is shown in equation (1).

$$\underset{\left(\bar{y}|_{\hat{z}}(s),(\hat{z};\bar{y}(0))\right)_j}{\arg\max} \left\{ \bar{q} z_j + \sum\nolimits_{s=1}^{S} \bar{p}(s)\, y_j|_{\hat{z}}(s) : \begin{array}{l} \bar{q} \sum\limits_{i=1}^{m} \bar{z}_i(j) \geq \bar{q} \hat{z}_j = \bar{p}(0) y_j(0) \\ y_j|_{\hat{z}}(s) \in Y_j|_{\hat{z}}(s) \end{array} \quad s \in S \right\}, \tag{1}$$

## 2.2 The consumers

Consumers play the same role in this production model as in the classical GEI model with production. They invest into firms because they want to transfer wealth between future uncertain states of nature, and to smooth out consumption across states of nature. Each consumer $i \in \{1, ..., m\}$ purchases stocks $z_i$ at stock price $q$ in period one in return for a dividend stream in the next period. The consumer's optimization problem is to maximize utility subject to a sequence of $(S+1)$ budget constraints. Each consumer $i \in \{1, ..., m\}$ is characterized by the set of standard assumptions for smooth economies introduced in Debreu [1].

**Assumption 3 (Smooth Utilities)** *For each consumer $i \in \{1, ..., m\}$, preferences are represented with smooth utilities defined on the strictly positive orthant, $X_i = \mathbb{R}^{l(S+1)}_{++}$.*

[1] All assumptions and their implications are introduced in Stiefenhofer (2009) in great detail.

Denote consumer $i$'s sequence of $(S+1)$ budget constraints

$$B_{z_i} = \left\{ (x_i, z_i) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R}^n_+ \middle| \begin{array}{c} p(0) \centerdot (x_i(0) - \omega_i(0)) = -qz_i \\ p(s)\Box(x_i(s) - \omega_i(1)) = \Pi(p_1, \Phi|_Z)\theta_i(z_i) \end{array} \right\}, \quad (2)$$

where $\theta_i$ in (2) denotes the endogenously determined ownership structure of consumer $i \in \{1, ..., m\}$, a $(n \times 1)$ vector defined by the mappings

$$\theta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+ \text{ for all } j \in \{1, ..., n\}, \quad (3)$$

where $z_i(j) \in R_+$ is a non-negative real number for every $j \in \{1, ..., n\}$. $\theta_{ij} = z_i(j)[\sum_i z_i(j)]^{-1}$ is the proportion of total payoff of financial asset $j \in \{1, ..., n\}$ hold by consumer $i \in \{1, ..., m\}$ after trade at the stock market took place in period one. $\Pi$ represents the payoff matrix of the economy of order $(S \times n)$.

The sequential optimization problem of the consumer $i \in \{1, ..., m\}$ is to invest into firms in period one in order to smooth out future uncertain consumption and to optimize consumption of goods in every $(S+1)$ spot market. For a given price system $p = (p(0), p(1), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$ of consumption goods and price system $q \in \mathbb{R}^n_{++}$ of financial assets (stocks), a consumer chooses bundles of consumption goods and quantities of stocks $(x, z)_i \in X_i \times \mathbb{R}^n_+$ such that $u_i(x_i; z_i)$ is maximized subject to the sequence of $(S+1)$ constraints in $B_{z_i}$. Algebraically, each $i \in \{1, ..., m\}$

$$(\overline{x}_i; \overline{z}_i) argmax \Big\{ u_i(x_i; z_i) : z_i \in \mathbb{R}^n_+, x_i \in B_{z_i} \Big\}. \quad (4)$$

## 2.3 Equilibrium equations

We introduce following prize normalization $\mathcal{S} = \{p \in R^{l(S+1)}_{++} : \|p\| = \Delta\}$ such that the Euclidean norm vector of the spot price system $p$ is a strictly positive real number $\mathbb{R}_{++}$. A competitive equilibrium of the production economy defined by the initial resource vector $\omega \in \Omega$ is a price pair $(p, q) \in \mathcal{S} \times \mathbb{R}^n_{++}$ if equality between demand and supply of physical goods and financial assets is satisfied in all states of nature, $s = 0, 1, ..., S$. Its associated competitive equilibrium allocation is a collection of vectors $[(x, z), (y, \widetilde{z})] \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}^{l(S+1)n} \times \mathbb{R}^n_-$ of consumption, production and financial quantities. Market clearance conditions are determined by the aggregate excess demands for physical goods and for financial assets as expressed by the equilibrium equations:

$$\begin{array}{ll} \text{(i)} & \sum_{i=1}^m (\overline{x}_i(0) - \omega_i(0)) = \sum_{j=1}^n \overline{y}_j(0) \\ \text{(ii)} & \sum_{i=1}^m (\overline{x}_i - \omega_i(1)) = \sum_{s=1}^S \sum_{j=1}^n \overline{y}_j(s) \\ \text{(iii)} & \sum_{j=1}^n \sum_{i=1}^m (\overline{z}_i)_j = 0, \; \sum_{i=1}^m \theta(\overline{z}_i)_j = 1 \text{ for all } j \in \{1, ..., n\} \end{array} \quad (5)$$

Stiefenhofer [7] shows that equilibria for this stock market model always exist.

# 3 Result

**Theorem 1** *For every producer $j \in \{1, ..., n\}$, the period two net activity vector $y_j(s)$ in available production set $Y_j|_Z(s)$ is independent of any present value vector $\beta_i$ for all $i \in \{1, ..., m\}$, and $s \in \{1, ..., S\}$.*

**Proof 1** *Using assumption* (1) *and assumption* (2), *let* $q_j(i) = \sum_{s=1}^{S} \beta_i(s) [p(s) \cdot y_j(s)]$, *where* $\beta_i$ *denotes* $i$*'s marginal evaluation of one additional unit of future income for* $\beta_i(s) \neq \beta_{i'}(s) \Leftarrow \beta_i \hat{\Pi} = 0$ *at* $S > n$, *for all* $i \in \{1, ..., m\}$, *where* $\hat{\Pi}(p_1, q, y) = \begin{bmatrix} -q_1 & \cdots & -q_n \\ p(1) \cdot y_1(1) & \cdots & p(1) \cdot y_n(1) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p_1(S) \cdot y_n(S) \end{bmatrix}$ *represents the full payoff matrix of order* $((S+1) \times n)$.

*Denote* $z_i(j)$ *the consumer's* $i \in \{1, ..., m\}$ *demand of quantity of stocks* $j \in \{1, ..., n\}$ *evaluated at* $q_j(i)$ *and* $\beta_i$.

*Now, let*

$$\begin{aligned} &\sum\nolimits_{i=1}^{m} \bar{z}_i(j) = \bar{z}_j \text{ for all } j \in \{1, ..., n\} \Rightarrow \bar{q} \\ &\sum\nolimits_{i=1}^{m} (\bar{x}_i(0) - \omega_i(0)) = \sum\nolimits_{j=1}^{n} \bar{y}_j(0) \Rightarrow \bar{p}(0) \\ &\sum\nolimits_{i=1}^{m} (\bar{x}_i(s) - \omega_i(1)) = \sum\nolimits_{j=1}^{n} \bar{y}_j(s), \quad \forall s \in S \Rightarrow \bar{p}(s) \end{aligned}$$

*Consider* $t = 0$ *optimization problem for* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., n\}$. *For given equity prices system* $\bar{q}$ *and spot prices system* $\bar{p}(0)$, *let each consumer* $i \in \{1, ..., m\}$

$$\underset{x(0) \in B_{z_i}, z_i}{max} \; u_i(x_i; z_i) \Rightarrow \bar{x}_i(0), \bar{z}_i$$

*and each producer* $j \in \{1, ..., n\}$

$$\underset{z(j)}{max} \; \bar{q}z(j) : \bar{q}z(j) = \bar{q} \sum\nolimits_{i=1}^{m} \bar{z}_i(j) \Rightarrow \bar{z}(j).$$

*Given maximum quantity of stocks producer* $j \in \{1, ..., n\}$ *can sell on the stockmarket,* $\bar{z}(j)$, *the problem of the producer* $j \in \{1, ..., n\}$ *is then to purchase capital* $y_j(0)$. *The problem of produce* $j \in \{1, ..., n\}$ *is then to maximize the level of capital at given spot price system* $\bar{p}(0)$, *and given cash acquired by issuing stocks,* $\bar{q}\bar{z}(j) = \bar{M}_j$. *Let*

$$\underset{y(0)_j}{max} \; \bar{p}(0) y_j(0) : \bar{M}_j = \bar{p}(0) y_j(0) \Rightarrow \bar{y}_j(0).$$

*Let the level of capital* $\bar{y}_j(0)$, *at financial policy* $\bar{z}(j)$ *determine a correspondence* $\Phi|_z$. *This correspondence maps* $\mathbb{R}^m_-$ *into* $\mathbb{R}^n_+$ *for every state of nature* $s \in \{1, ..., S\}$. *This correspondence describes the production set available to the firm in period two, denoted* $Y_j\,|_{\bar{z}}(s)$ *for all* $s \in \{1, ..., S\}$. *Consider* $t = 1$ *optimization problem for* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., n\}$. *For given* $\bar{p}(s)$, *each consumer* $i \in \{1, ..., m\}$

$$\underset{x(s) \in B_{z_i}}{max} \; u_i(x_i; \bar{z}_i) \Rightarrow \bar{x}_i(s),$$

*and each producer* $j \in \{1, ..., n\}$, *given his endogenized production set* $Y_j\,|_{\bar{z}}(s)$ *for all* $s \in \{1, ..., S\}$

$$\underset{y_j(s)}{max} \{\bar{p}(s) \cdot y_j(s) : y_j(s) \in Y_j\,|_{\bar{z}}\,(s)\} \Rightarrow \bar{y}_j(s).$$

*We have constructed a sequential two argument linear objective function with endogenous asset structure*

$$\underset{(\bar{z}, \bar{y}(s), \bar{y}(0))_j}{\arg\max} \left\{ \bar{q}z_j + \sum\nolimits_{s=1}^{S} \bar{p}(s) \cdot y_j(s) \;\middle|\; \begin{array}{l} y_j(s) \in Y_j|_z\,(s) \\ \bar{q} \sum_{i=1}^{m} \bar{z}_i(j) \geq \bar{q}z_j = \bar{p}(0) y_j(0) \quad \forall s \in S \end{array} \right\}, \tag{6}$$

*which is independent of $\bar{\beta}_j$, and consequently the choice of $y_j$ is independent of all* $i \in \{1, ..., m\}$.

The result shows, by construction, the independence of the objective function of the firm from the utility of the stock holders. This decentralizes the objective function of the firm, and introduces a new class of objective functions, where firms maximize long run profits.

# 4 Conclusion

The paper considers a property of the objective function of the model of the firm introduced in Stiefenhofer [7]. It shows the separation of economic decisions of the agents in an incomplete markets environment. This follows from endogenizing the asset structure. The result generalizes the profit maximization criterion of the Arrow-Debreu model to an economic scenario, where incomplete markets are present, and the asset structure endogenized.

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